



Universal Relations for Elastic Hybrid Stars and Quark Stars

Chun Ming Yip^{1,2}, **Shu Yan Lau**³, and **Kent Yagi**⁴

¹ Department of Physics and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong SAR, People's Republic of China

² GSI Helmholtzzentrum für Schwerionenforschung, Planckstraße 1, D-64291 Darmstadt, Germany

³ eXtreme Gravity Institute, Department of Physics, Montana State University, Bozeman, Montana 59717, USA

⁴ Department of Physics, University of Virginia, Charlottesville, Virginia 22904, USA

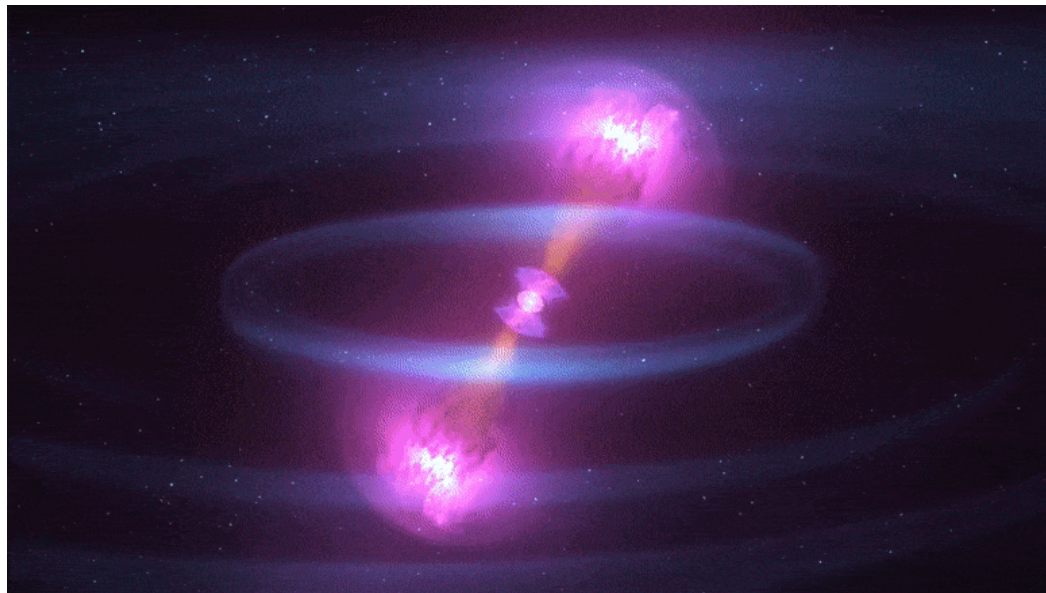
Phys. Rev. D 114, 043024 (arXiv:2510.15149)

Wroclaw University of Science and Technology

09.09.2026

My Current Position

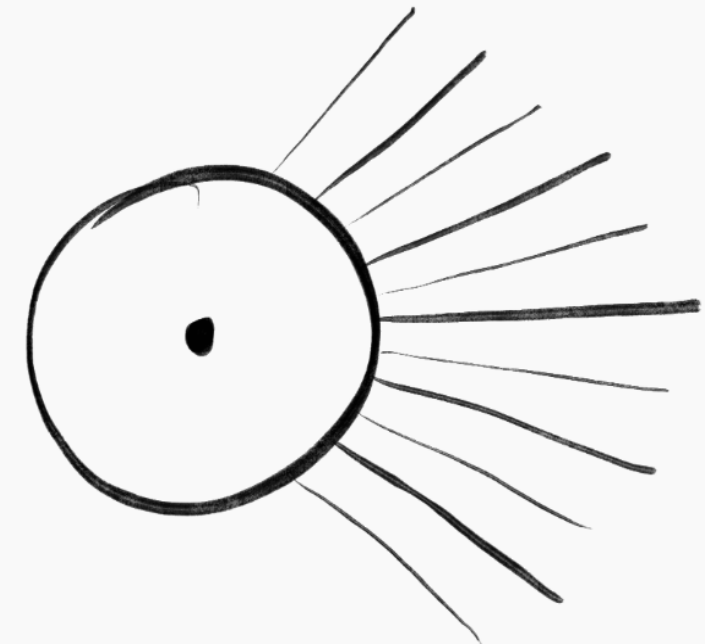
- PhD student at GSI Helmholtz Centre for Heavy Ion Research and Heidelberg University in Germany (supervisor: Andreas Bauswein)
- Member of **HEAVYMETAL** collaboration



Animation of binary neutron star merger

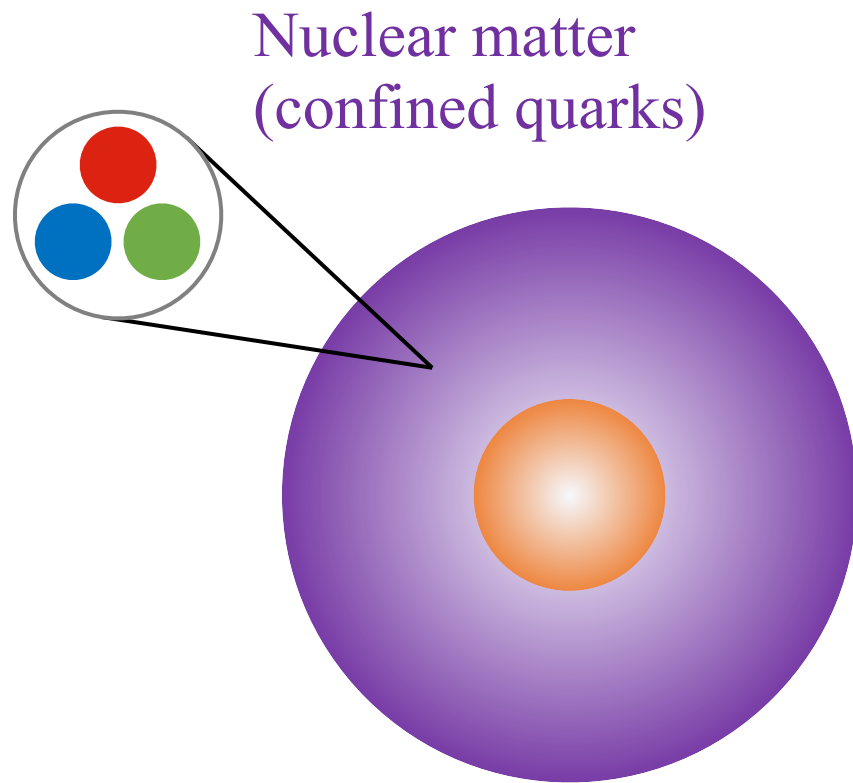
The origin of the heavy elements

HEAVYMETAL is a project to uncover the nuclear and astrophysical pathways leading to the creation of the rapid neutron capture elements

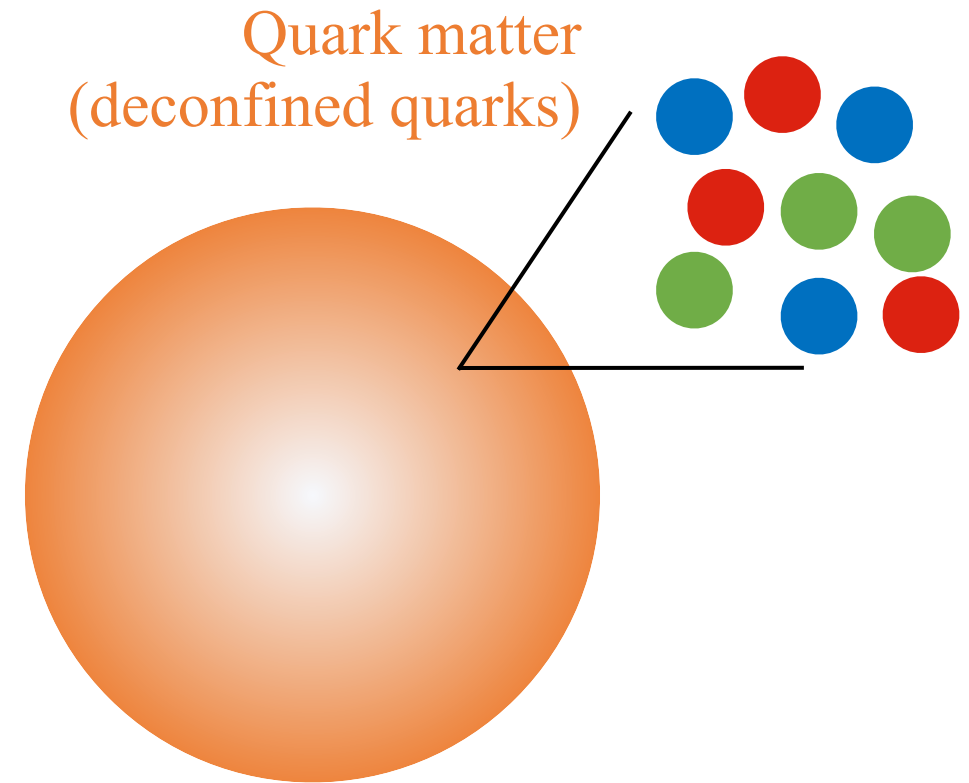


Neutron Star Interior

- Deconfined quark matter may exist in neutron stars
- Hybrid stars (quark stars) are neutron stars partially (entirely) consisting of quark matter



Hybrid star model



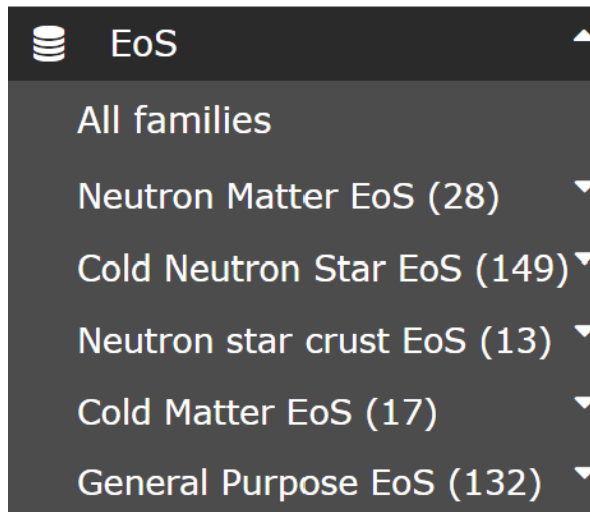
Quark star model

Equation of State (EOS)

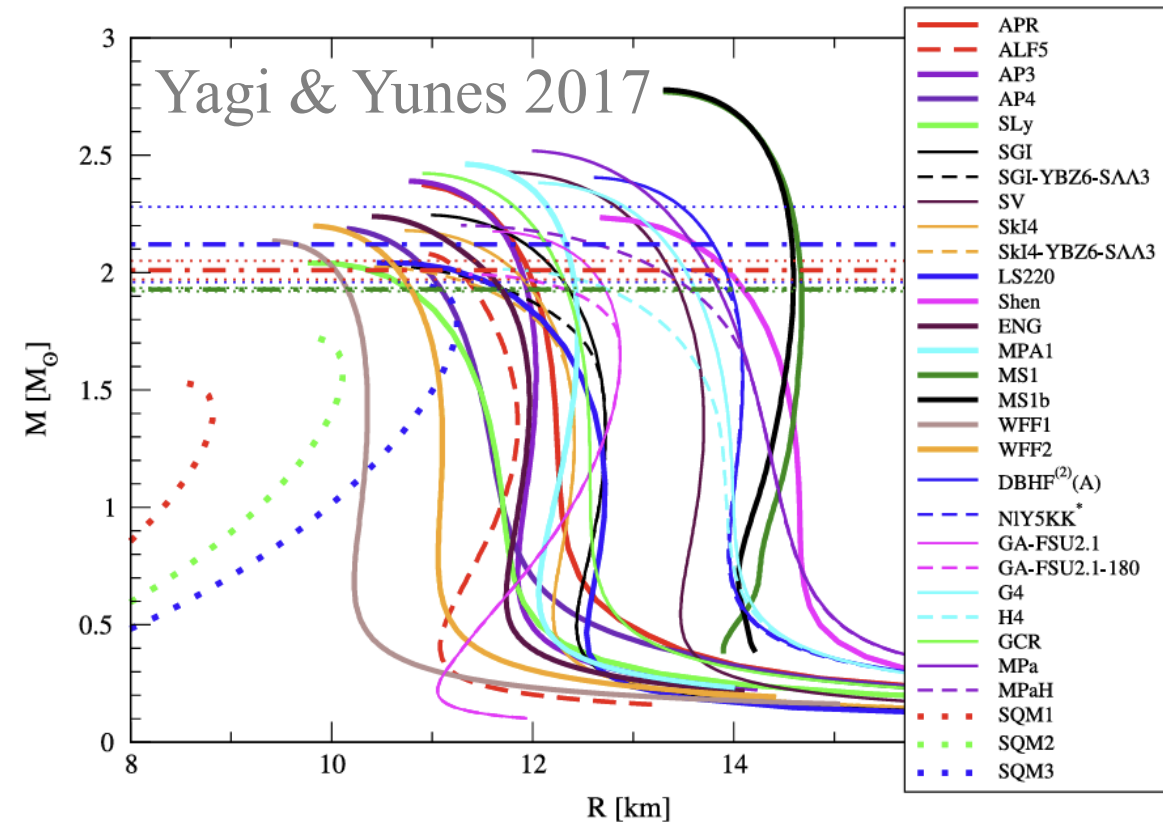
- Describe the relation between pressure p and energy density ρ
- With an EOS, we can calculate the mass M and radius R of a static neutron star

Equation of State (EOS)

- Describe the relation between pressure p and energy density ρ
- With an EOS, we can calculate the mass M and radius R of a static neutron star
- The correct EOS, however, is still unknown...



A large number of EOS
available on CompOSE



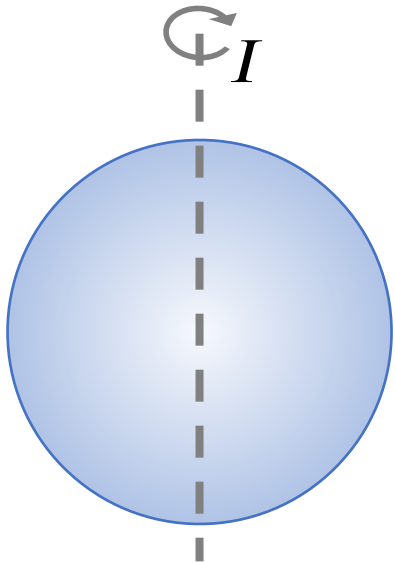
Different EOS predict
distinct stellar structures

Multipole Moments

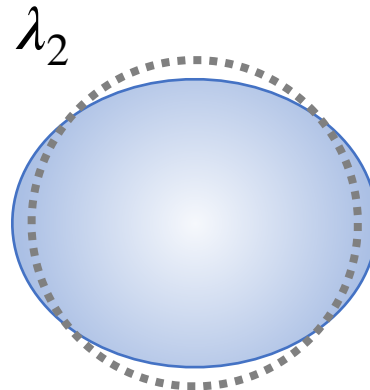
- Neutron stars are not static because of
 - spin
 - external gravitational field

Multipole Moments

- Neutron stars are not static because of
 - spin
 - external gravitational field
- By perturbation analysis, we can also calculate

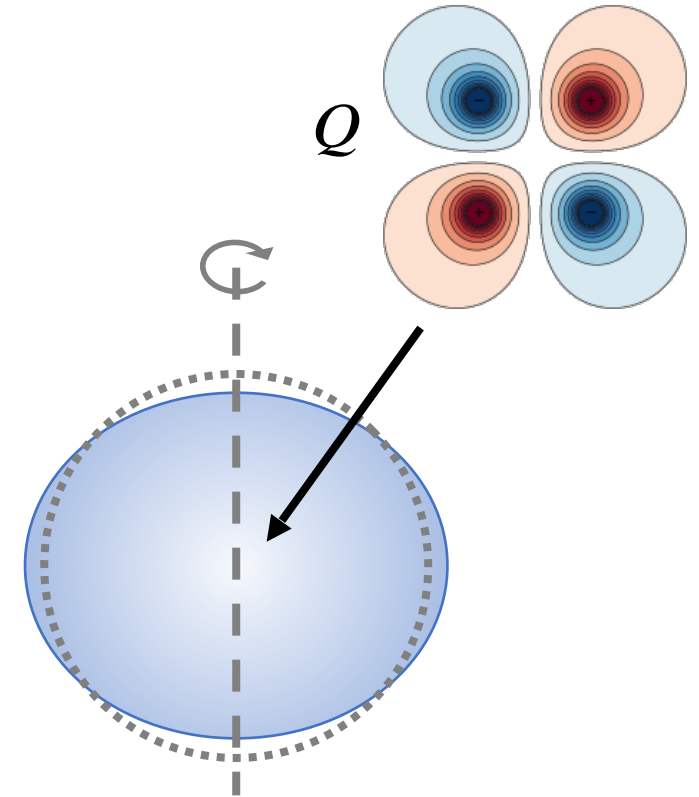


Moment of inertia



Tidal deformability

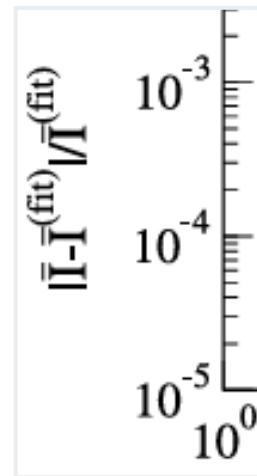
External
gravitational field



Spin-induced quadrupole moment

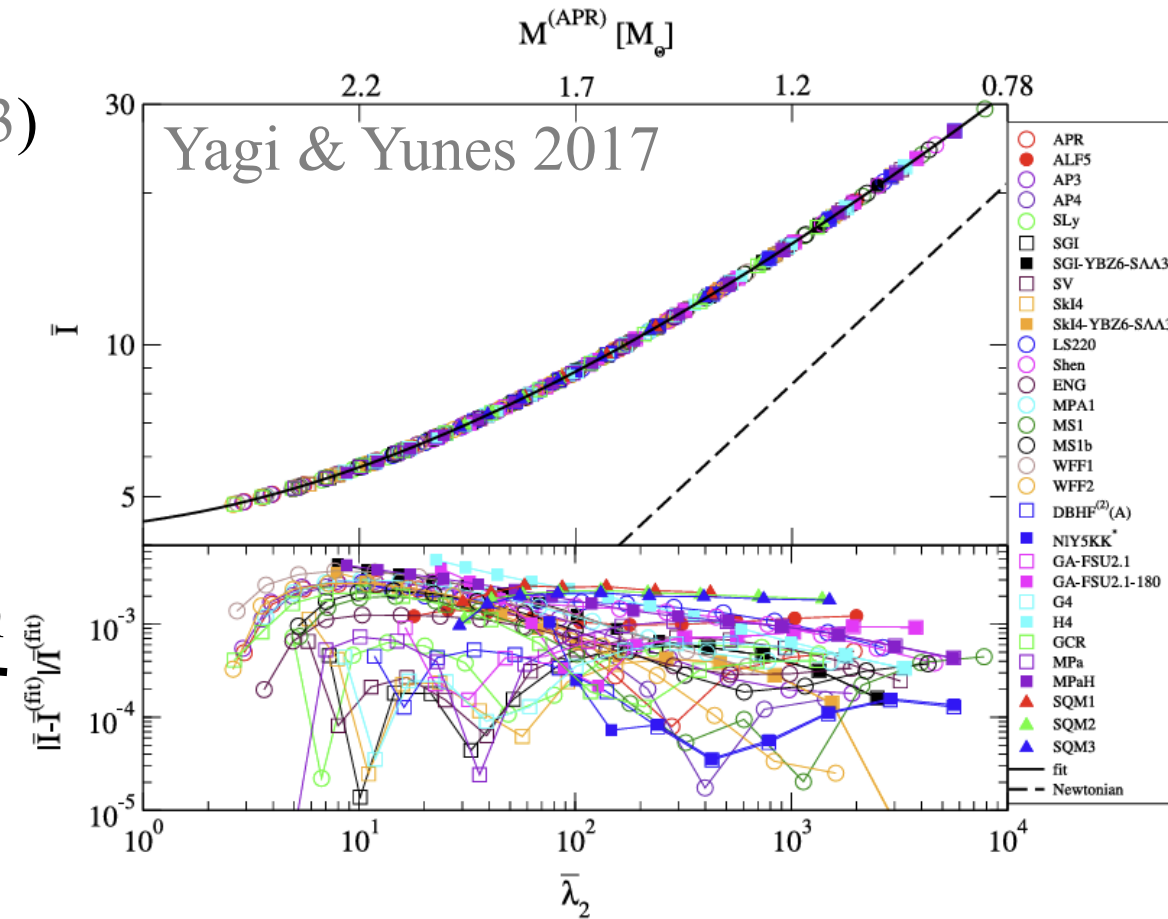
Universal Relations

- The correlation between stellar properties can be robust
- I - λ_2 - Q relations are insensitive to EOS variation within $\sim 1\%$ for neutron stars (Yagi & Yunes 2013)



Absolute deviation
from fitting relation

Zoom-in



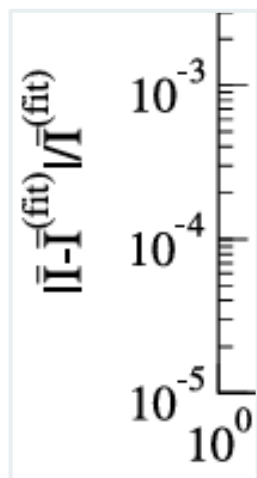
But different EOS predict
very similar I - λ_2 relations!

Universal Relations

- The correlation between stellar properties can be robust
- I - λ_2 - Q relations are insensitive to EOS variation within $\sim 1\%$ for neutron stars (Yagi & Yunes 2013)
- Assumption: the stars are made of perfect fluid with *isotropic* pressure

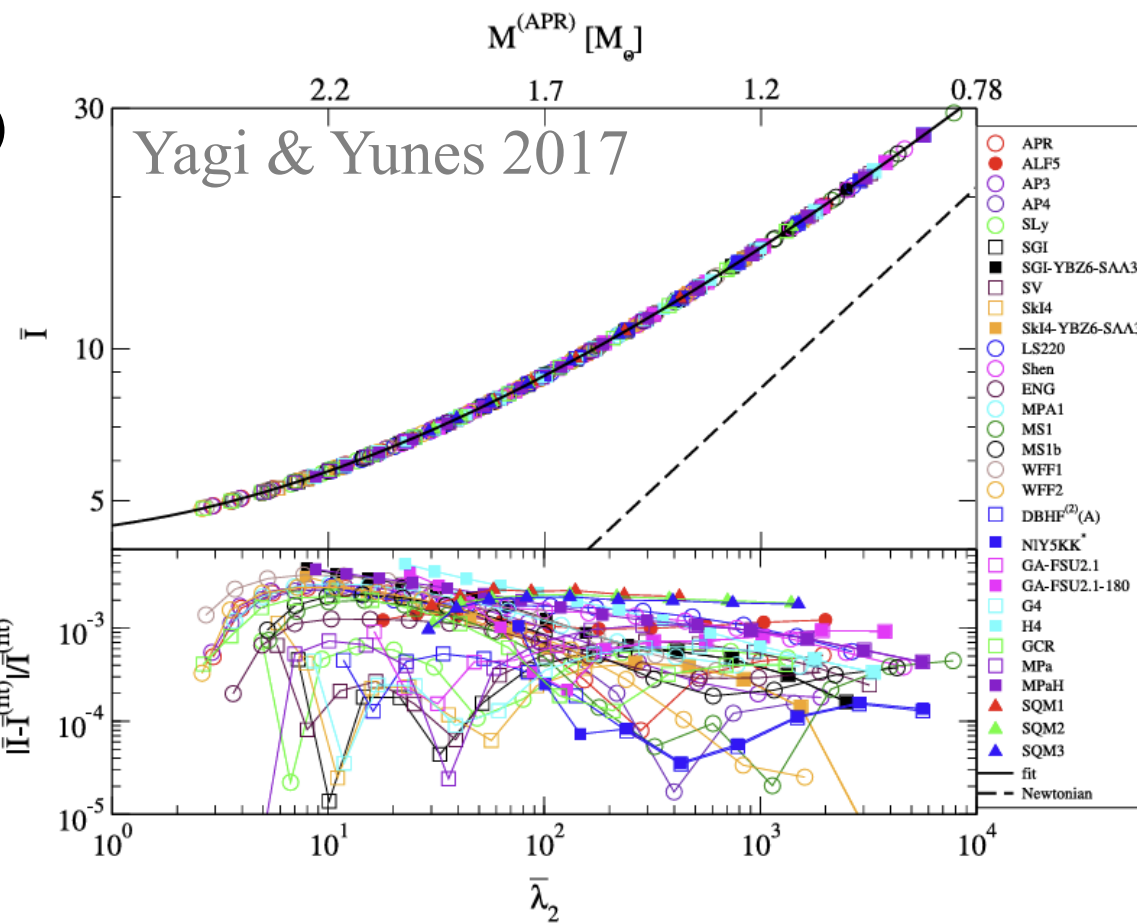
Stress-energy tensor

$$\begin{array}{c}
 t \quad r \quad \theta \quad \varphi \\
 \left[\begin{array}{cccc}
 -\rho & 0 & 0 & 0 \\
 0 & p & 0 & 0 \\
 0 & 0 & p & 0 \\
 0 & 0 & 0 & p
 \end{array} \right]
 \end{array}$$



Zoom-in

Absolute deviation
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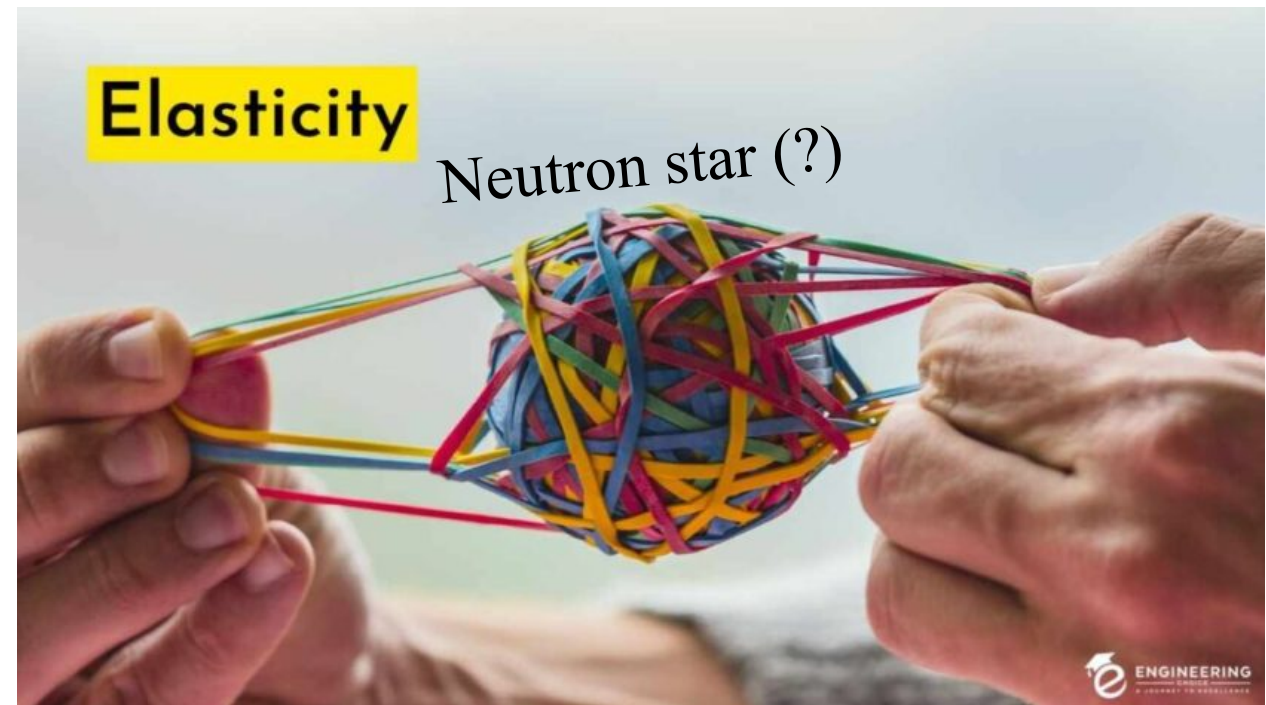
But different EOS predict
very similar I - λ_2 relations!

What If Neutron Stars are Elastic?

- Pressure inside neutron stars can become *anisotropic*
- To calculate their I - λ_2 - Q relations, we adopt
 - elastic star models (Karlovini & Samuelsson 2003, Dong *et al.* 2025)
 - perturbation equations with pressure anisotropy (Yagi & Yunes 2015)

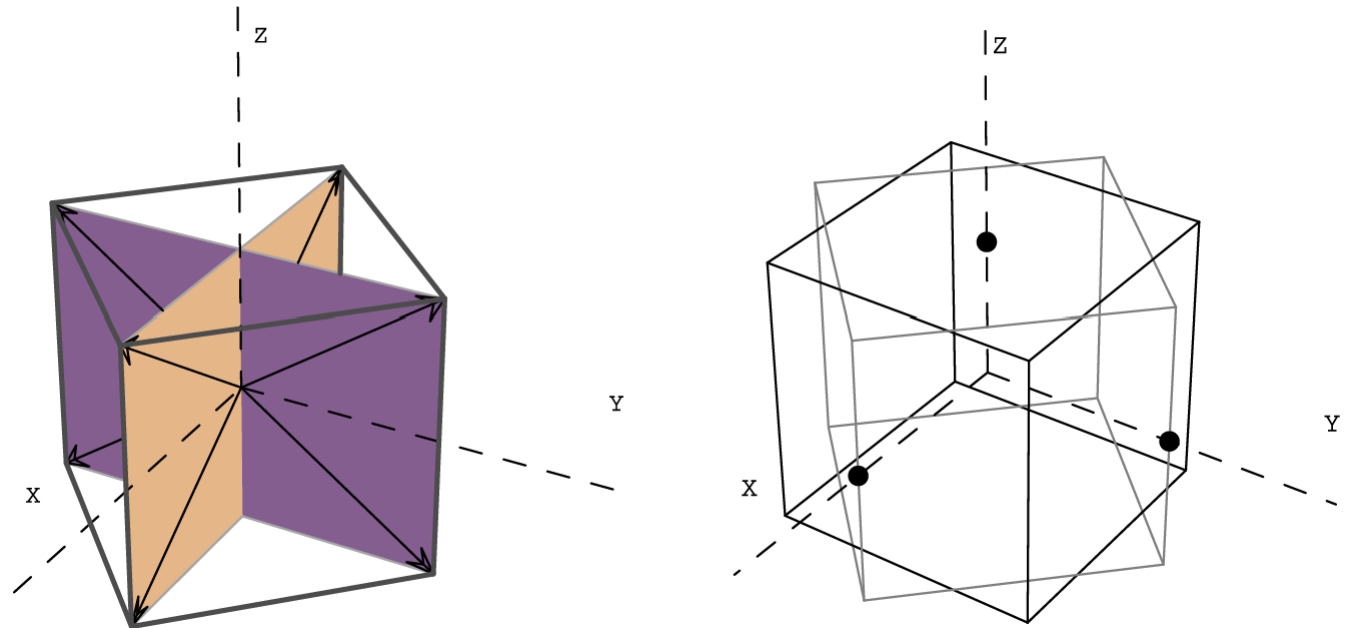
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- To calculate their I - λ_2 - Q relations, we adopt
 - elastic star models (Karlovin & Samuelsson 2003, Dong *et al.* 2025)
 - perturbation equations with pressure anisotropy (Yagi & Yunes 2015)
- But wait... are hybrid stars and quark stars indeed elastic?



Crystalline Color Superconducting (CCS) Quark Matter

- Assume the quark matter is CCS
- Have a rigidity $\sim 1,000$ times larger than that of neutron star crust (Mannarelli *et al.* 2007)

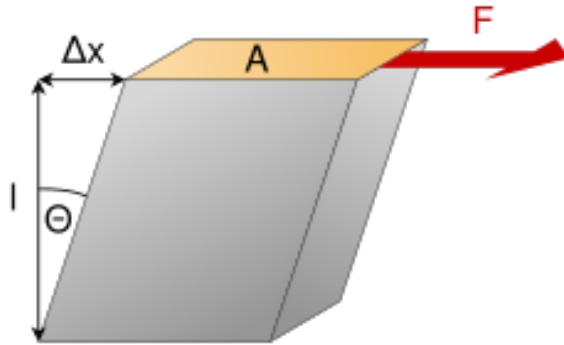


Crystal structure of quark matter (Mannarelli *et al.* 2007)

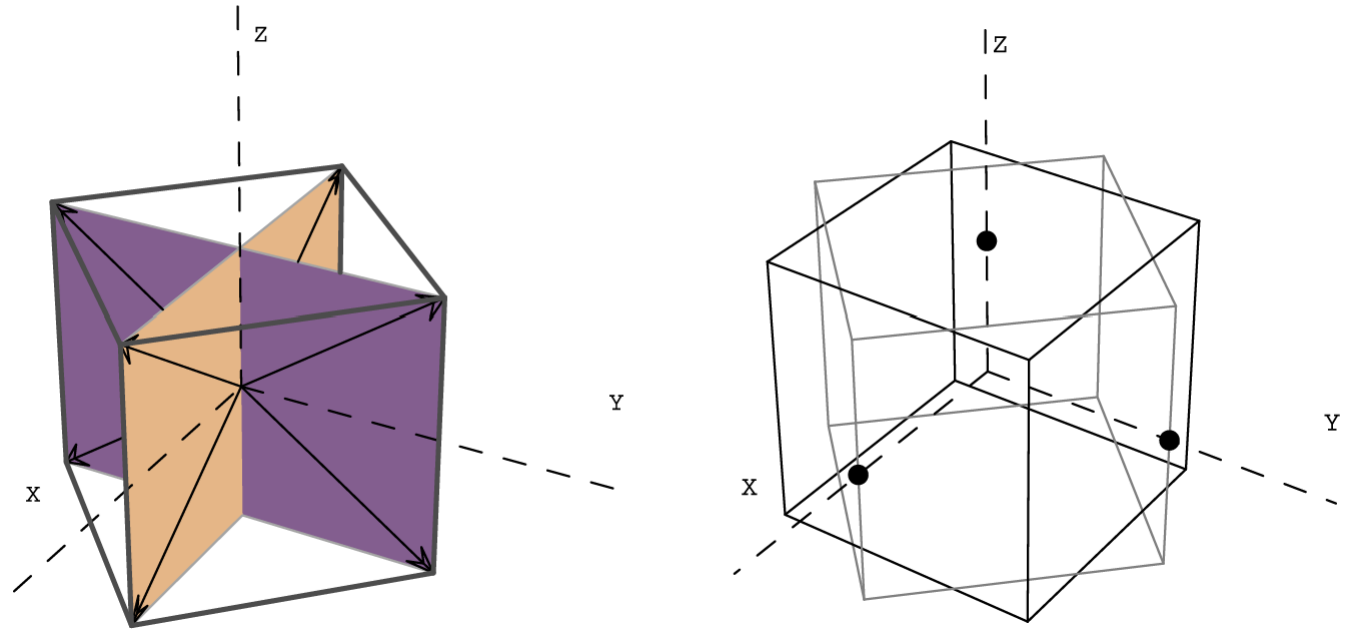
Crystalline Color Superconducting (CCS) Quark Matter

- Assume the quark matter is CCS
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- Shear modulus: $\tilde{\mu} = \kappa \sqrt{\rho}$
 κ is at most $7 \times 10^{26} \text{ cm}^{1/2} \text{ g}^{1/2} \text{ s}^{-2}$



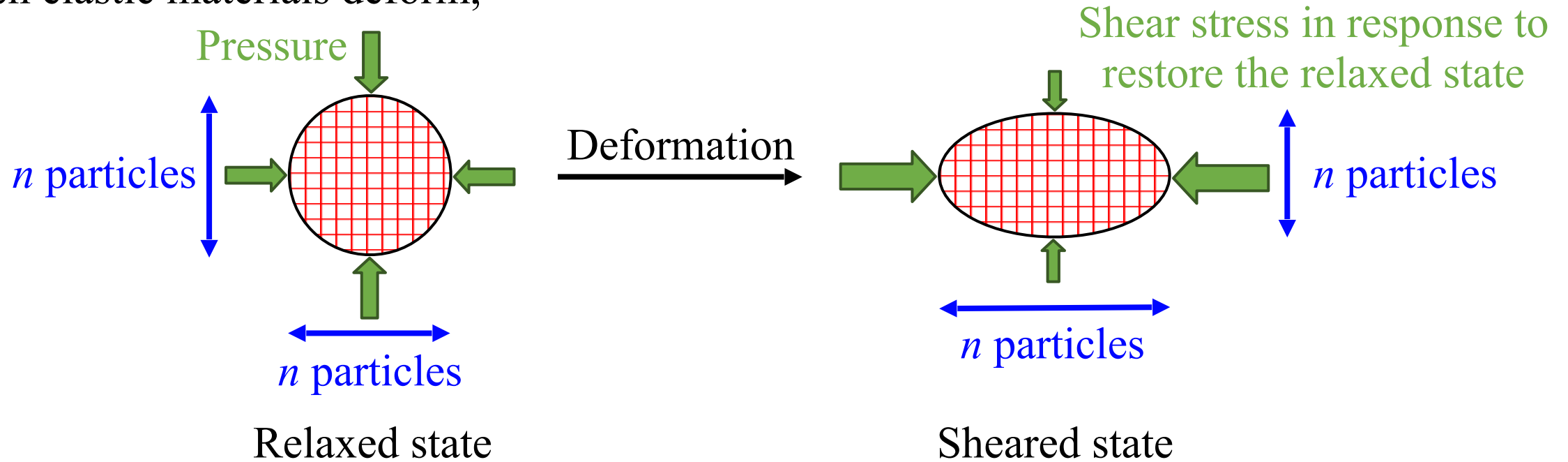
$$\tilde{\mu} \equiv \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta x/l}$$



Crystal structure of quark matter (Mannarelli *et al.* 2007)

Elastic Materials

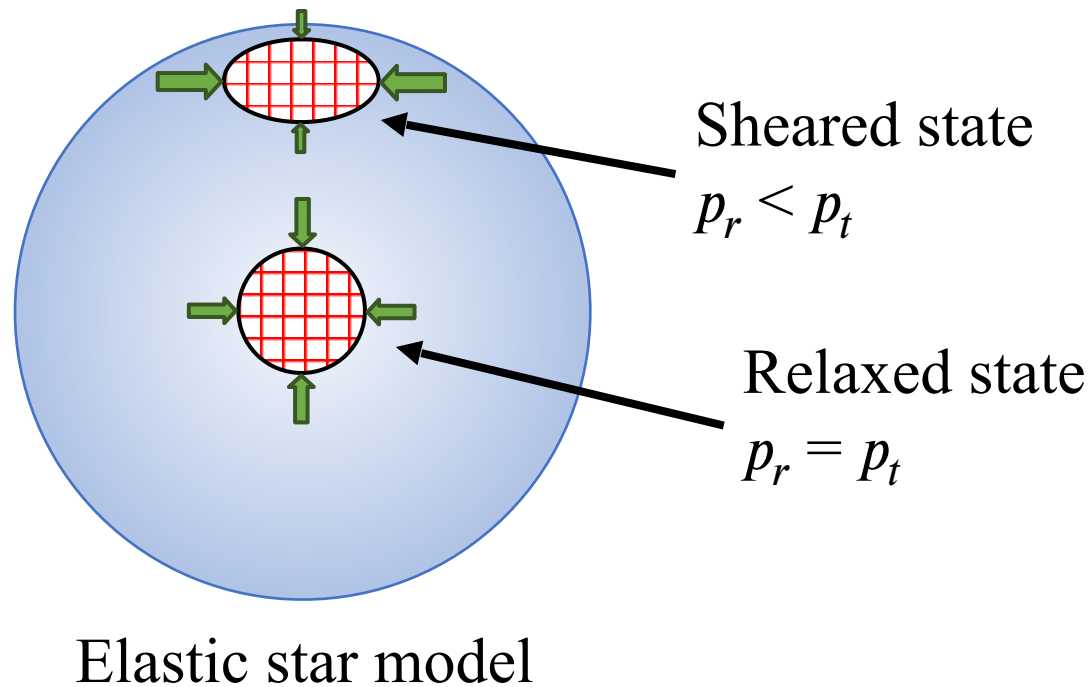
- When elastic materials deform,



- Sheared state*: the **particle number density** has directional dependence

Elastic Star Models

- Remain spherically symmetric in shape without perturbation
- However, $p_r \neq p_t$ because the particle number density in radial and tangential directions are different



Stress-energy tensor

t	r	θ	φ
$-\rho$	0	0	0
0	p_r	0	0
0	0	p_t	0
0	0	0	p_t

Radial pressure

Tangential pressure

Elastic Star Models

- A two-component EOS is adopted:

$$\begin{aligned}\rho &= \tilde{\rho} + \tilde{\mu} S^2 \\ p_r &= \tilde{p} + \left(\tilde{\Omega} - 1 \right) \tilde{\mu} S^2 + \frac{2}{3} \sigma_0\end{aligned}$$

- **Unsheared component**: fluid contribution
- **Sheared component**: additional shear contribution from the CCS quark matter (formulated by Karlovini & Samuelsson 2003)

Elastic Star Models

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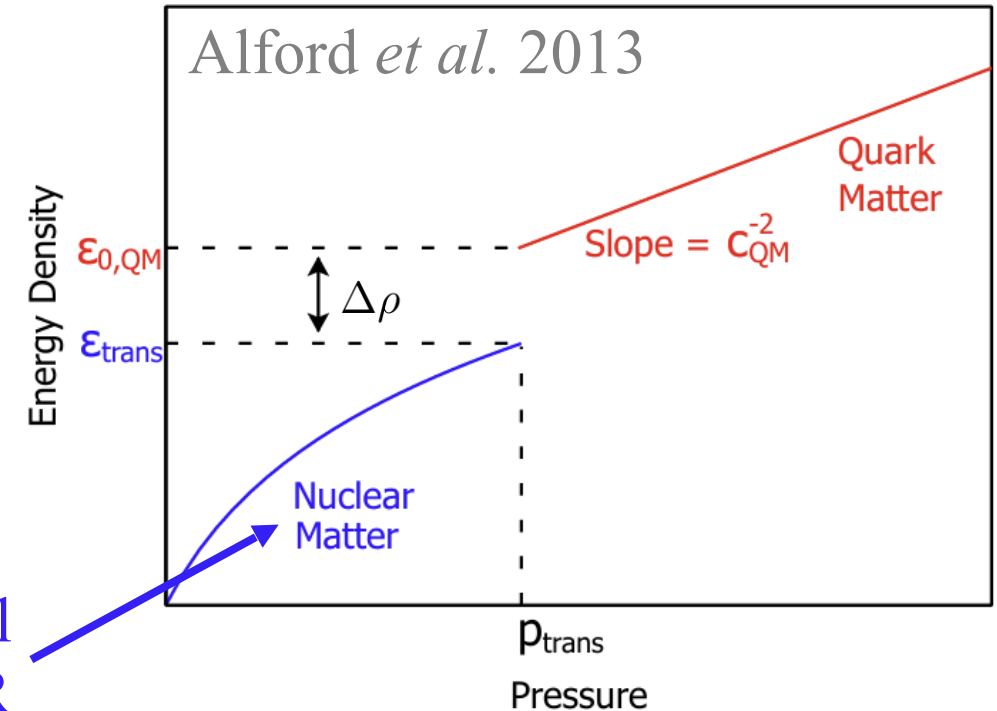
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- Unsheared component:** fluid contribution
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Pick a nuclear model
e.g., APR

Quantified by three parameters
 $\{p_{\text{trans}}, \Delta\rho, c_{\text{QM}}^2\}$



$\tilde{\rho}$ versus $\tilde{p}$ of a sample hybrid EoS
(constant sound speed parametrization)
as the **unsheared component**

Workflow

1st step: construct an elastic star model (Karlovin & Samuelsson 2003, Dong *et al.* 2025)

$$\frac{dm}{dr} = 4\pi r^2 \rho,$$

$$\frac{d\tilde{p}}{dr} = \frac{\tilde{\beta}}{r\beta_r} \left\{ -(\rho + p_r) \frac{4\pi p_r r^3 + m}{r - 2m} - 2\sigma_0 \right. \\ \left. + 4 \left(e^{\lambda/2} z - 1 \right) \left[\tilde{\mu} + 3\tilde{\mu} S^2 + \frac{\sigma_0}{2} (\tilde{\Omega} - 1) \right] \right\},$$

$$\frac{dz}{dr} = \frac{z}{r} \left[\frac{r}{\tilde{\beta}} \frac{d\tilde{p}}{dr} - 3 \left(e^{\lambda/2} z - 1 \right) \right],$$

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2nd step: solve the perturbation equations (Yagi & Yunes 2015)

$$\frac{d^2\omega_1}{dr^2} = 4 \frac{\pi r^2 (\rho + p_r) e^\lambda - 1}{r} \frac{d\omega_1}{dr} \\ + 16\pi (\rho + p_r - \sigma_0) e^\lambda \omega_1,$$

$$\frac{dh_2}{dr} = -\frac{4\pi p_r r^3 - m + r}{r} e^\lambda \frac{dk_2}{dr} + \frac{3}{r} e^\lambda h_2 + \frac{2}{r} e^\lambda k_2 + \frac{1 + 8\pi p_r r^2}{r^2} e^{2\lambda} m_2 + \frac{r^3}{12} e^{-\nu} \left(\frac{d\omega_1}{dr} \right)^2$$

$$+ 4\pi (\rho + p_r) \frac{4\pi p_r r^3 + m}{r} e^{2\lambda} \xi_2 + 8\pi e^\lambda \sigma_0 \xi_2,$$

$$\frac{dk_2}{dr} = -\frac{dh_2}{dr} - \frac{4\pi p_r r^3 + 3m - r}{r^2} e^\lambda h_2 + \frac{4\pi p_r r^3 - m + r}{r^3} e^{2\lambda} m_2,$$

$$\frac{d\xi_2}{dr} = \frac{r - 2m}{6r [(\rho + p_r) (4\pi p_r r^3 + m) + 2(r - 2m) \sigma_0]} \left\{ -6r^2 (\rho + p_r) \frac{dh_2}{dr} - 12\sigma_0 r^2 \frac{dk_2}{dr} \right. \\ \left. - 3 \left[r^2 (\rho + p_r) \frac{d^2\nu}{dr^2} - 4\sigma_0 \right] \xi_2 - 12r\sigma_2^{(2)} + 2r^3 (\rho + p_r - \sigma_0) e^{-\nu} \omega_1 \left[\left(r \frac{d\nu}{dr} - 2 \right) \omega_1 - 2r \frac{d\omega_1}{dr} \right] \right\}.$$

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3rd step: extract I , λ_2 , and Q

$$\omega_1^{\text{ext}} = \Omega \left(1 - \frac{2I}{r^3} \right)$$

$$h_2^{\text{ext}} = -\frac{Q}{r^3} + \mathcal{O} \left(\frac{1}{r^4} \right)$$

$$\lambda_2 = \frac{2}{3} k_2^{\text{tid}} R^5$$

2nd step: solve the perturbation equations (Yagi & Yunes 2015)

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$$\left. + 4 \left(e^{\lambda/2} z - 1 \right) \left[\tilde{\mu} + 3\tilde{\mu} S^2 + \frac{\sigma_0}{2} \left(\tilde{\mu} - \frac{\lambda}{2} \right) \right] \right\},$$

$$\frac{dz}{dr} = \frac{z}{r} \left[\frac{r}{\tilde{\beta}} \frac{d\tilde{p}}{dr} - 3 \left(e^{\lambda/2} z - 1 \right) \right],$$

Input: $\kappa, \{p_{trans}, \Delta \rho, c_{QM}^2\}$, and nuclear model
Output: I, λ_2 , and Q

ns (Yagi & Yunes 2015)

3rd step: extract I, λ_2 , and Q

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Stellar Structures are Modified

- Take $\{p_{\text{trans}}, \Delta \rho, c_{\text{QM}}^2\} = \{6 \times 10^{33} \text{ erg cm}^{-3}, 2 \times 10^{13} \text{ g cm}^{-3}, 0.33 \text{ c}^2\}$ and APR
- Quark matter rigidity provides extra effective pressure
- The maximum mass that can be supported increases

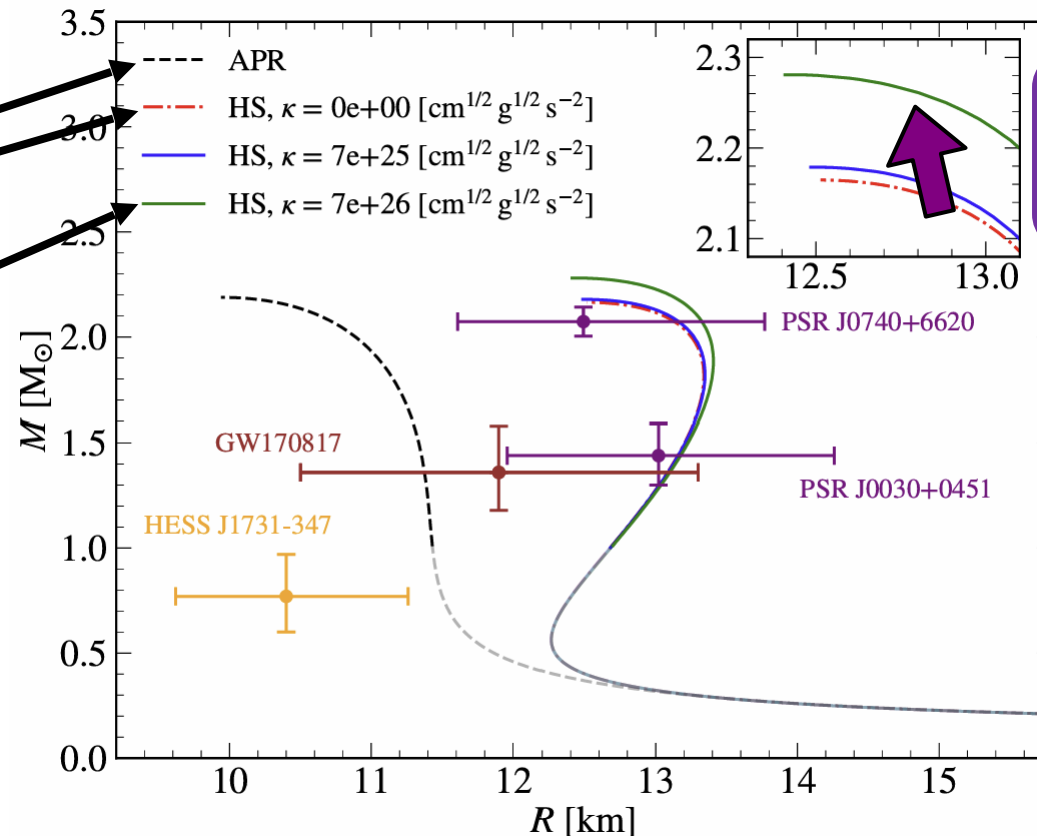
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Nuclear matter neutron star

Fluid hybrid star

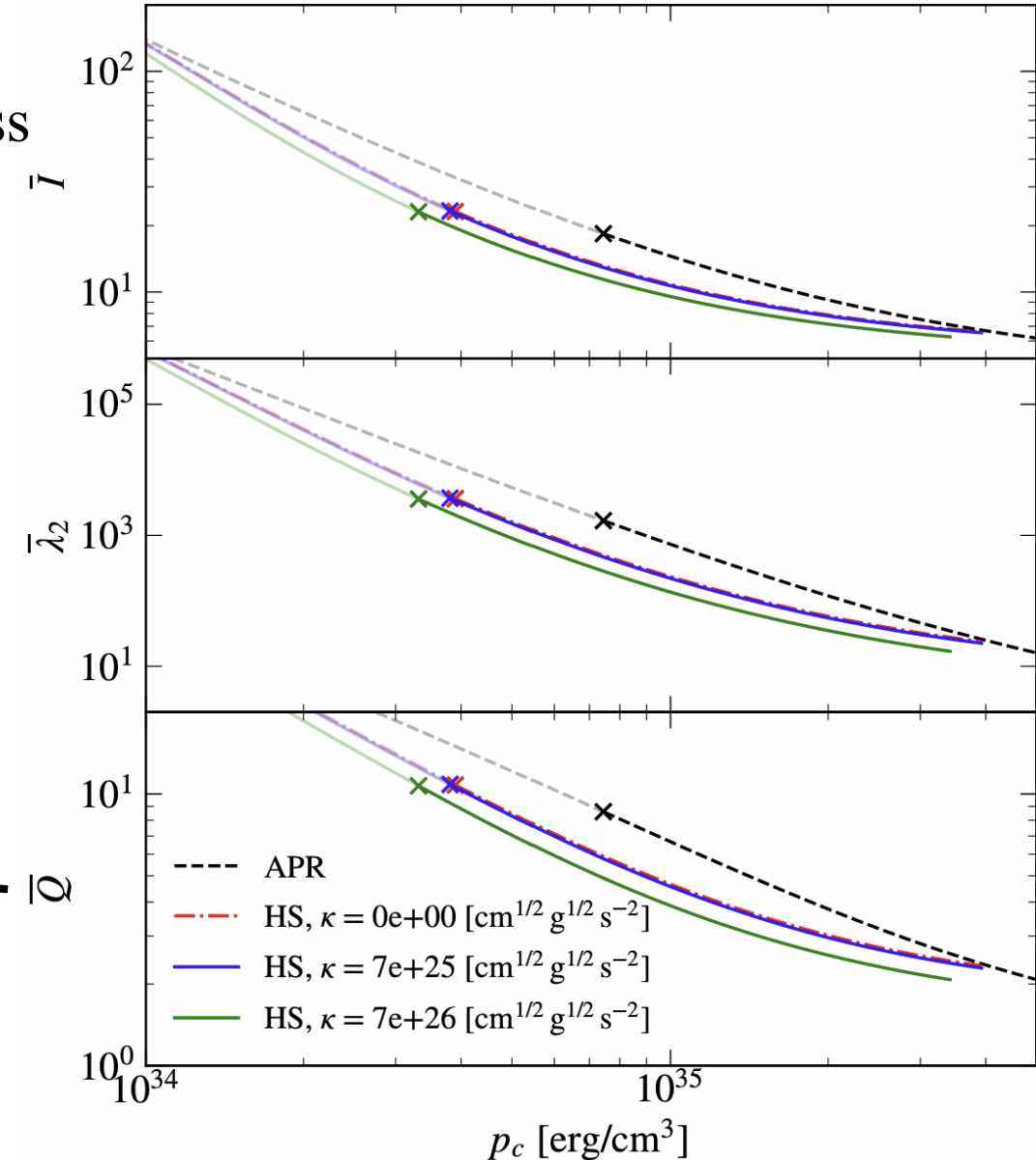
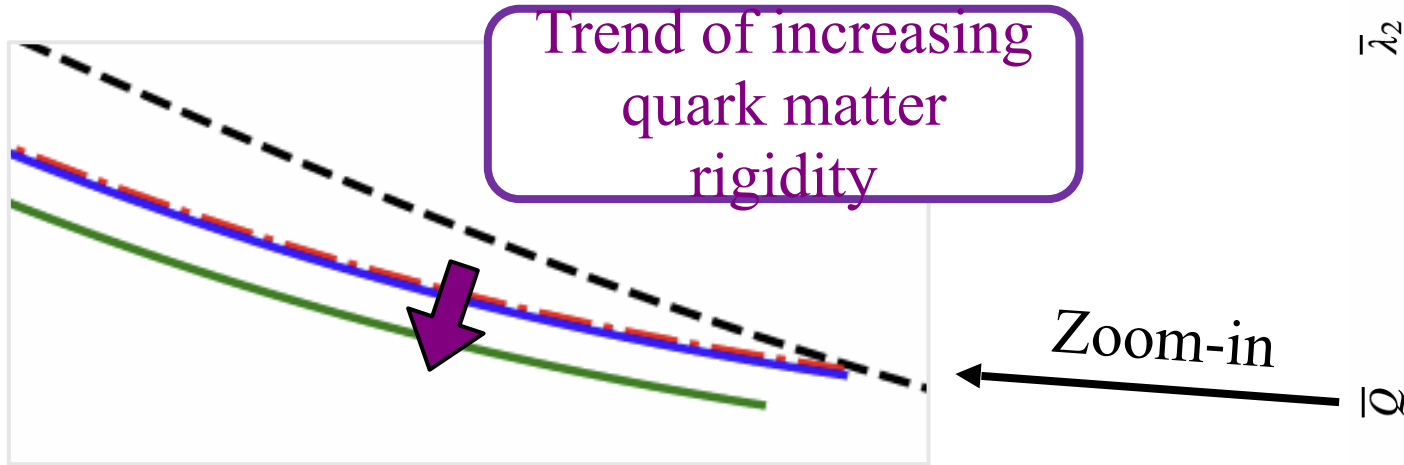
With maximal CCS
quark matter rigidity



Trend of increasing
quark matter
rigidity

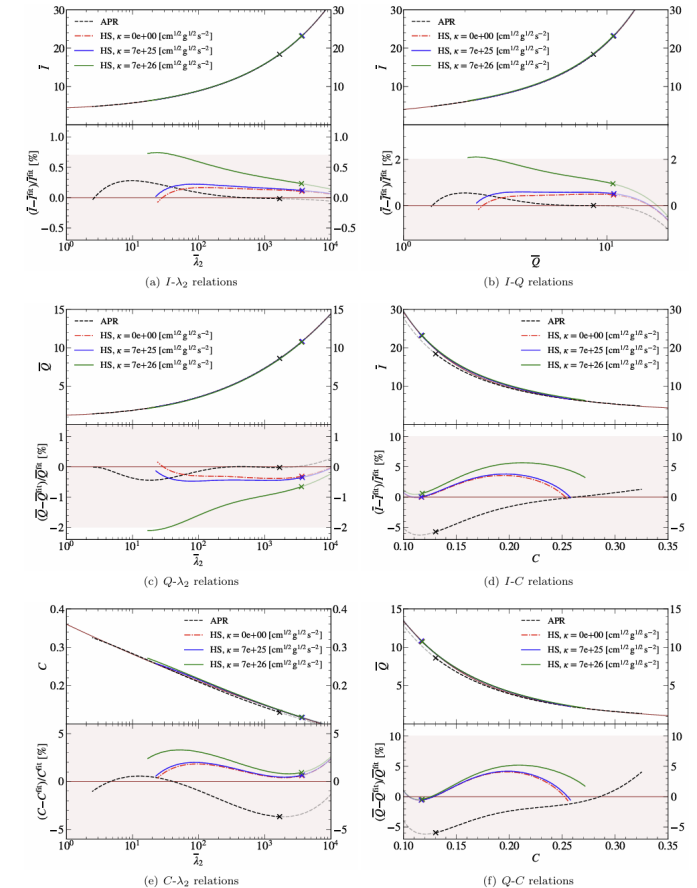
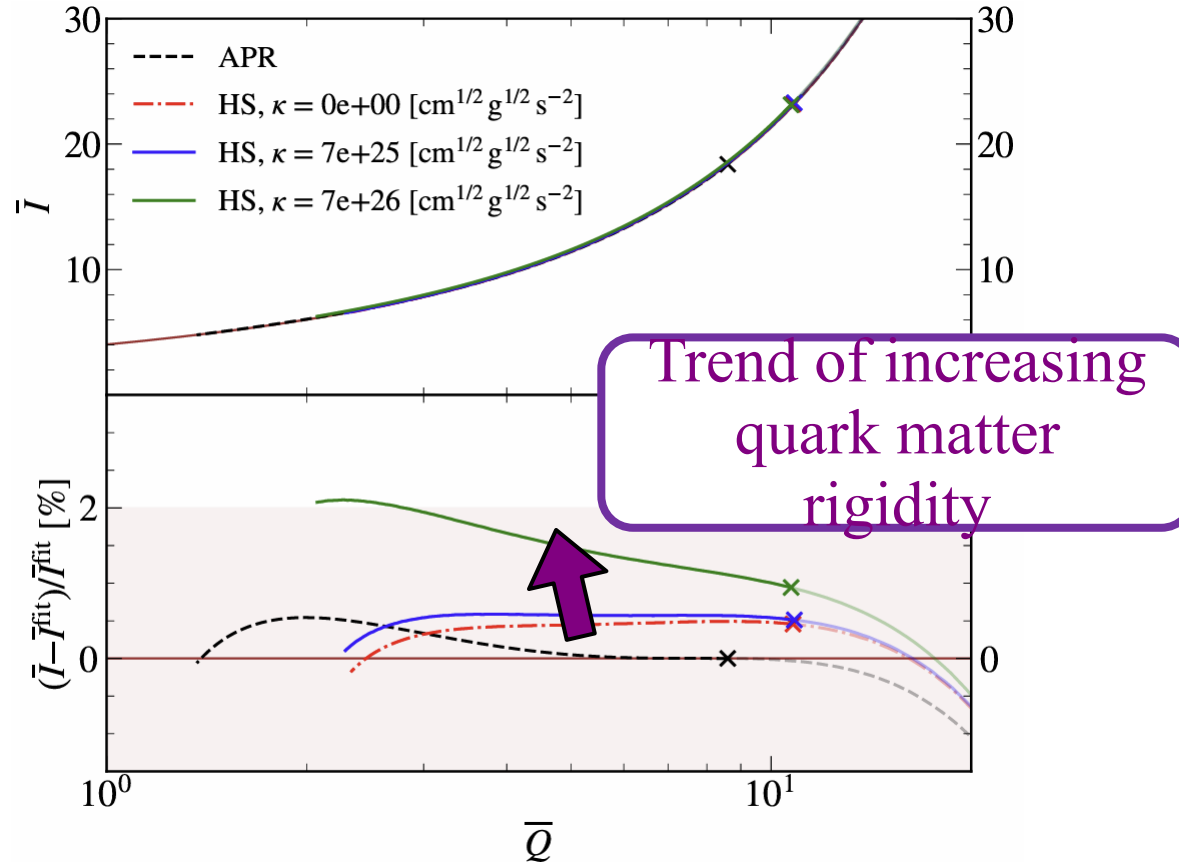
Multipole Moments are Modified

- For a hybrid star with fixed central pressure, larger quark matter rigidity $\rightarrow$ smaller dimensionless I , λ_2 , and Q
- How about their correlations?



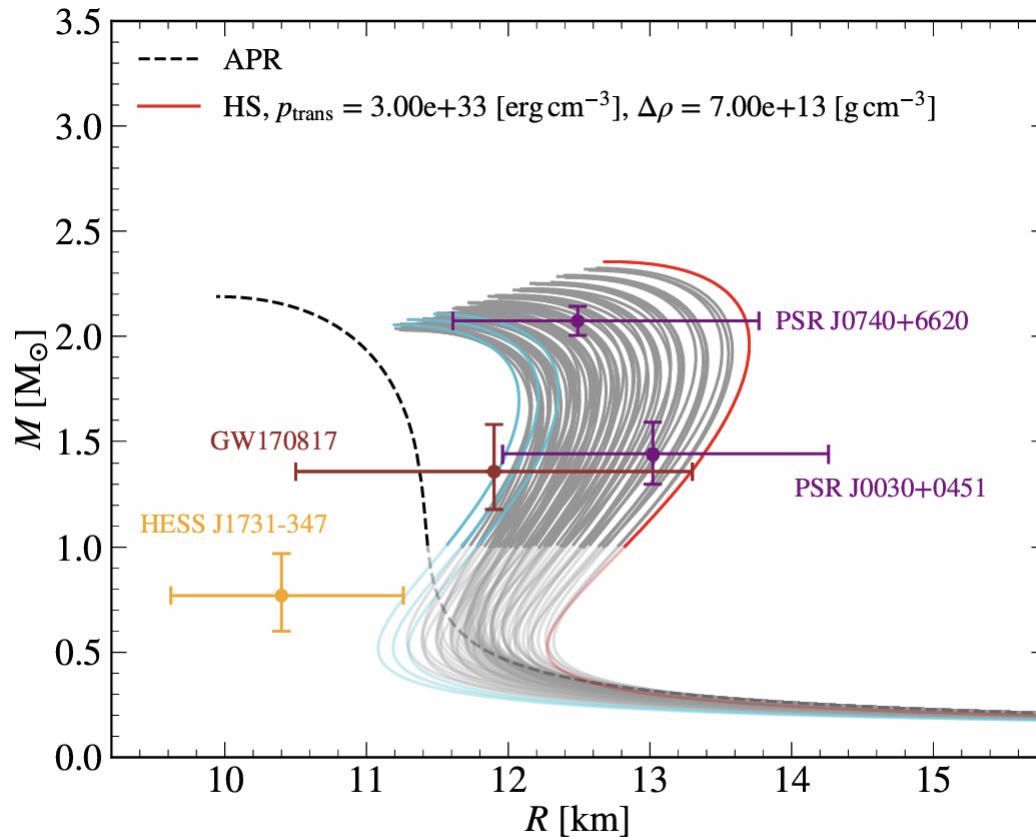
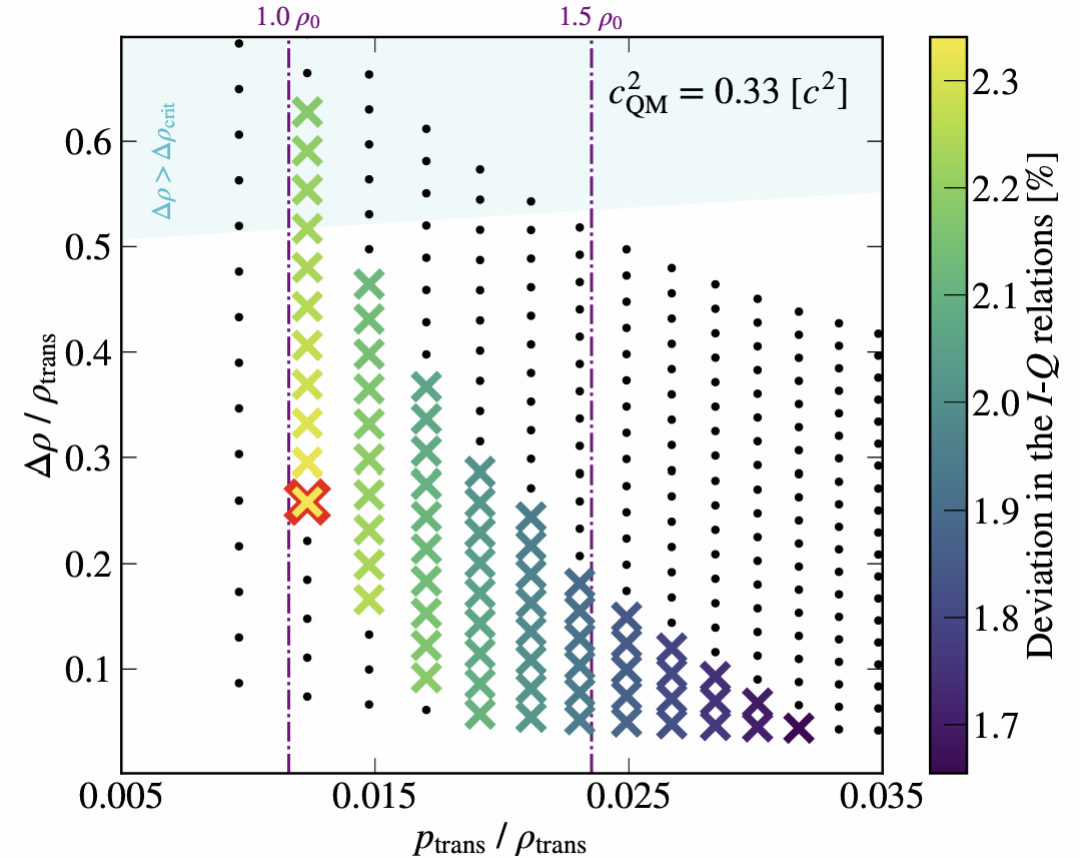
I - λ_2 - Q Relations are Less Robust

- Take the I - Q relations as examples
- Larger quark matter rigidity $\rightarrow$ larger deviation
- Maximum deviation from fitting relation $\sim 2\%$ (fluid models: $\sim 1\%$)



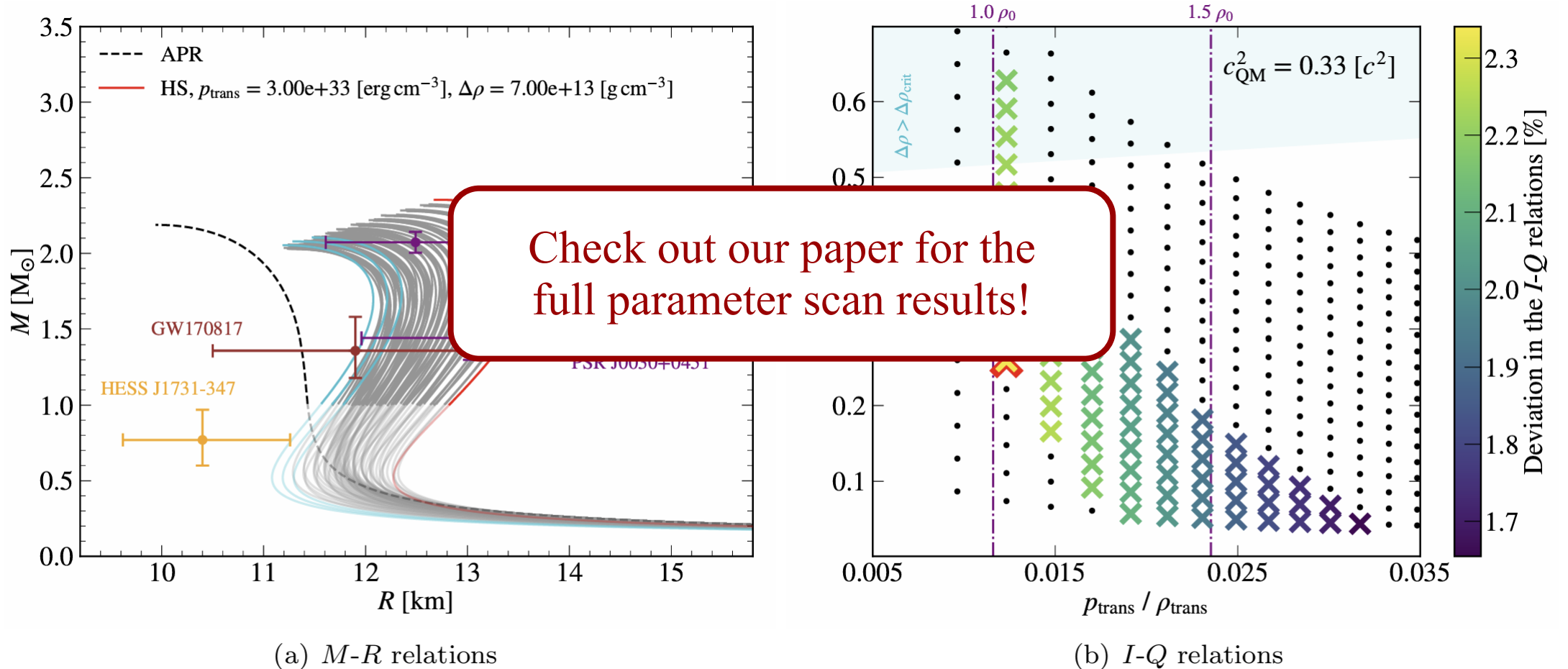
Parameter Scan - Hybrid Stars

- Varying $\{p_{\text{trans}}, \Delta\rho, c_{\text{QM}}^2\}$ and the nuclear model
- Maximum deviation $\sim 3\%$ for elastic hybrid stars (fluid models: $\sim 1\%$)

(a) M - R relations(b) I - Q relations

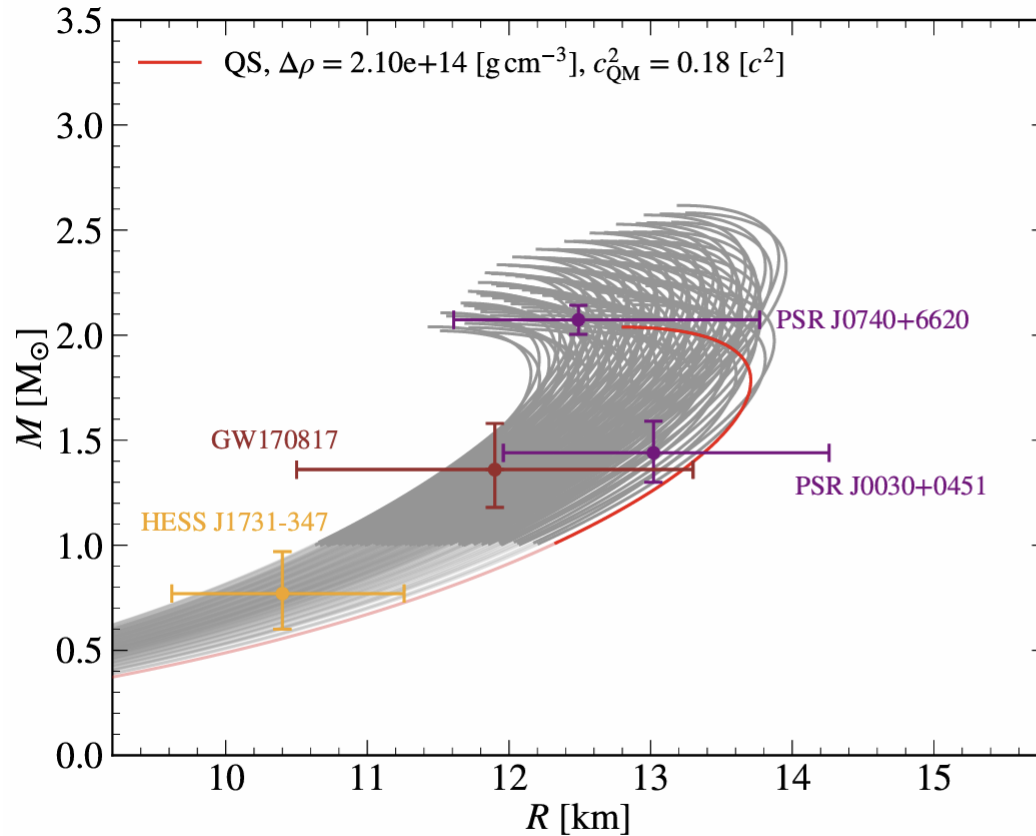
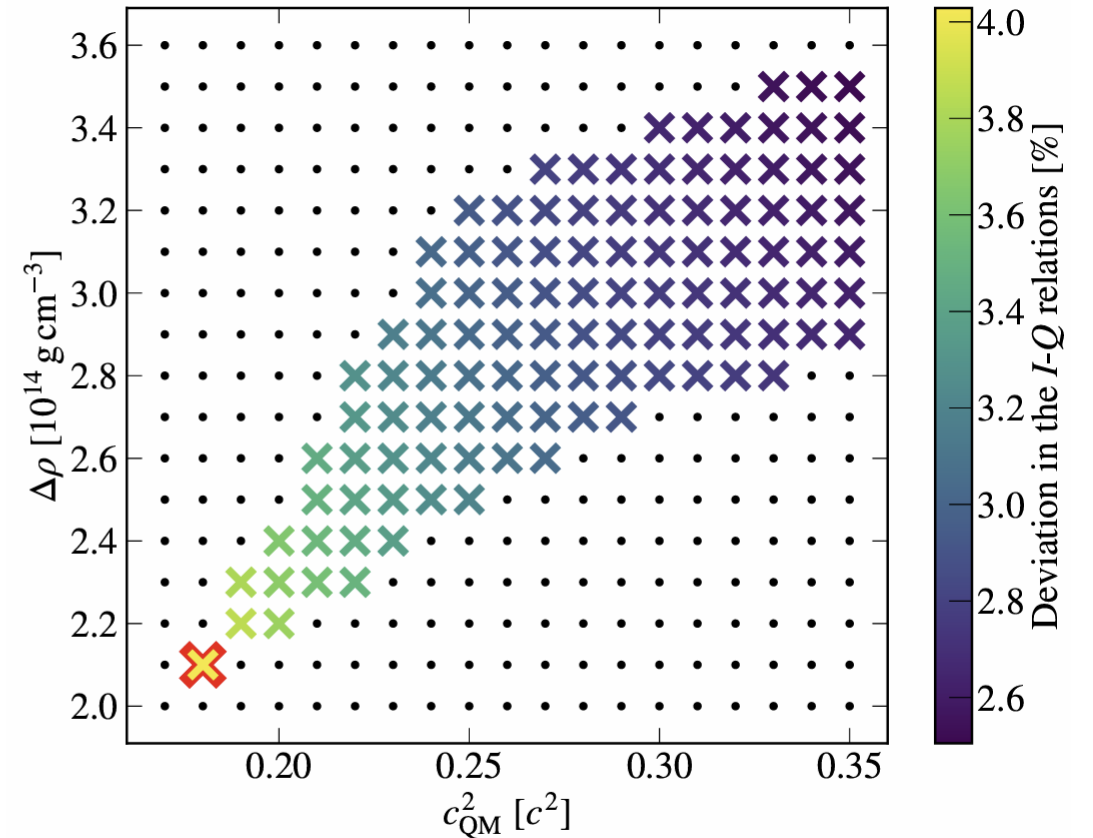
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Parameter Scan - Quark Stars

- Varying $\{ \Delta \rho, c_{\text{QM}}^2 \}$
- Maximum deviation $\sim 4\%$ for elastic quark stars (fluid models: $\sim 1\%$)

(a) M - R relations(b) I - Q relations

Possible Extension

- A more generic form of the stress-energy tensor under perturbation:

$$\begin{array}{cccc}
 t & r & \theta & \varphi \\
 \left[\begin{array}{cccc}
 -\rho & 0 & 0 & 0 \\
 0 & p_{11} & p_{12} & p_{13} \\
 0 & \dots & p_{22} & p_{23} \\
 0 & \dots & \dots & p_{33}
 \end{array} \right]
 \end{array}$$

- Stellar Pulsation of elastic hybrid stars and quark stars
- Numerical simulations
- Origin of I - λ_2 - Q universality

Related Work

Gravitational-wave constraints on neutron-star pressure anisotropy via universal relations

[Victor Guedes](#)^{1,*}, [Siddarth Ajith](#) ¹, [Shu Yan Lau](#) ², and [Kent Yagi](#) ¹

Show more ▾

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Universal relations for rapidly rotating cold and hot hybrid stars

[Noshad Khosravi Largani](#) , [Tobias Fischer](#) , [Armen Sedrakian](#) , [Mateusz Cierniak](#),
[David E Alvarez-Castillo](#), [David B Blaschke](#)

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Pages 3539–3554, <https://doi.org/10.1093/mnras/stac1916>

Published: 23 July 2022 **Article history** ▾

Broadening of universal relations at the birth and death of a neutron star

[Victor Guedes](#)^{1,*}, [Shu Yan Lau](#)¹, [Cecilia Chirenti](#) ^{2,3,4,5}, and [Kent Yagi](#) ¹

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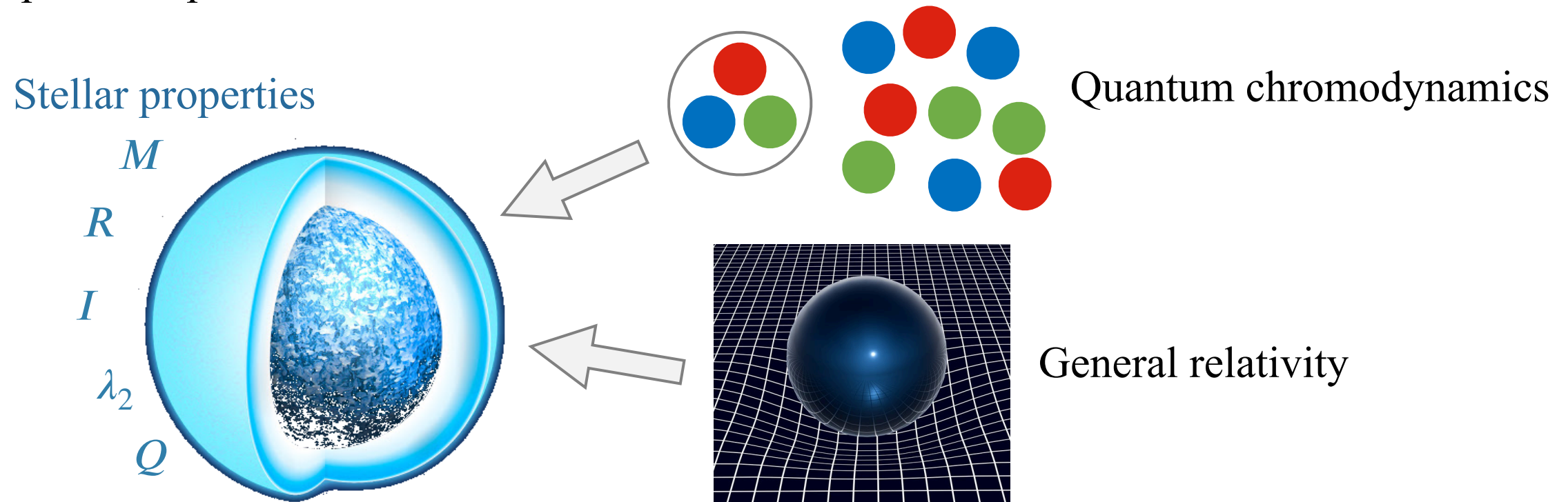
Gravitational wave detection

Rapid rotation and finite-temperature effects

Proto-neutron star in supernovae

Summary

- Stellar properties are affected by rigidity
- I - λ_2 - Q relations for elastic hybrid stars and quark stars are less robust
- A potential probe of neutron star interior



Thank You!